(8 pages) **Reg. No. :** ....

Code No. : 5416 **Sub. Code : WMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2025.

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. If the norm is defined as $\|x\|_p = \left(|x_1|^p + |x_2|^p\right)^{1/p}$, then the shape of closed unit sphere is spherical if ______

   (a) $p \geq 1$ (b) $p < 1$

   (c) $p = 0$ (d) $p = 2$

2. A linear transformation T is continuous if and only if T is ______

   (a) one-one (b) onto

   (c) bounded (d) isomorphism

3. The elements of the conjugate space of N are ______

   (a) Operators (b) Not bounded

   (c) Functionals (d) Isomorphisms

4. The mapping $T \to T^*$ from $\mathcal{B}(N)$ into $\mathcal{B}(N^*)$ is a ______ mapping

   (a) Linear (b) Continuous

   (c) Norm preserving (d) All the above

5. If $x \perp y$, then $\|x - y\|^2 =$ ______

   (a) $\|x\|^2$ (b) $\|x - y\|^2$

   (c) $\|y\|^2$ (d) $\|x\|^2 - \|y\|^2$

6. If $\{e_i\}$ is a complete ortho normal set in Hilbert space H, then the Fourier coefficients of an arbitrary vector x are ______

(a) $\Sigma|x,e_i|^2$ (b) $\Sigma(x,e_i)e_i$

(c) $|(x, ei)|^2$ (d) (x,ei)

7. Every self adjoint operator is ______

(a) Normal

(b) Unitary

(c) Both (a) and (b)

(d) Neither (a) nor (b)

8. $(\alpha T)^* =$ ______

(a) αT^* (b) αT

(c) $\bar{\alpha} T$ (d) $\bar{\alpha} T^*$

9. If N is a normal operator on H, then $\|N\|^2 =$ ______

(a) $\|N\|$ (b) $\|N^*\|$

(c) $\|N\|^2$ (d) $\|Nx\|$

10. $\det(T) \neq 0 \Leftrightarrow T$ is ______

(a) Singular (b) Linear

(c) Non-singular (d) Continuous

11. The characteristic equation of T is ______

(a) $\det(T-\lambda)=0$ (b) $\det(\lambda-TI)=0$

(c) $\det(\lambda-I)=0$ (d) $\det(T-\lambda I)=0$

12. The set of all eigen values of T is called ______ of T

(a) Conjugate (b) Adjoint

(c) Spectrum (d) Normal

13. The spectral radius $r(x)$ of x ______

(a) $=\|x\|$ (b) $>\|x\|$

(c) ≤ 0 (d) $\leq\|x\|$

14. The radical R of A is the intersection of all its ______

(a) Proper ideal (b) Minimal left ideals

(c) Maximal left ideals (d) Left singular

15. $\sigma(x^n) =$ ____________

(a) $n\sigma(x)$ (b) $\sigma(x)$

(c) $\sigma(x)^n$ (d) n

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove the following:

(i) Norm is a continuous function

(ii) Addition and scalar multiplication in norm are jointly continuous.

Or

(b) Check whether the mapping $x \to F_x$ from N into N^{**} is an isometric isomorphism or not.

17. (a) If B and B' are Banach spaces and if T is a linear transformation of B into B', then prove that T is continuous if and only if its graph is closed.

Or

(b) State and prove Bessel's inequality.

18. (a) Obtain a necessary and sufficient condition for an operator T on H to be unitary.

Or

(b) List the properties of the adjoint operation $T \to T^*$ on $\mathcal{B}$(H) and prove them.

19. (a) Define determinant of a matrix. Discuss its properties.

Or

(b) If T is normal, then prove that eigen spaces M_i's span H.

20. (a) Let I be a pro-per closed two-sided ideal in A. Then check whether quotient algebra A/I is a Banach algebra or not.

Or

(b) Show that the mapping $x \to x^{-1}$ of G into G is continuous and hence a homoeomorphism of G onto itself.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b)

21. (a) Define a norm for a coset $x+M$ in the quotient space N/M and hence prove N/M is a Banach space, if N is a Banach space.

Or

(b) State and prove Hahn Banach theorem.

22. (a) If M is a closed linear subspace of a Hilbert space H, then show that $H = M \oplus M^{\perp}$.

Or

(b) If $\{e_i\}$ is an ortho normal set in a Hilbert space H and if x is an arbitrary vector in H, then show that $x - \Sigma(x, e_i) \perp e_j$, for each j.

23. (a) Let H be a Hilbert space and let f be an arbitrary functional in H*. Check for the existence of a vector y in H such that $f(x) = (x, y)$ for every x in H. Prove its uniqueness if it exists.

Or

(b) If T is an operator on H, then prove the following

(i) $(T_x\ x) = 0$ for all $x \Rightarrow T = 0$

(ii) T is self adjoint $\Leftrightarrow (T_x, x)$ is real for all x

(iii) T is normal $\Leftrightarrow$ its real and imaginary parts commute.

24. (a) State and prove the spectral theorem.

Or

(b) Show that the spectral resolution of normal operator T is unique.

25. (a) Derive the formula for the spectral radius.

Or

(b) Define the spectrum of an element x in a Banach algebra A and prove that it is non-empty.

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(8 pages) **Reg. No. :** ....

Code No. : 5417 **Sub. Code : WMAM 42**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2025.

Fourth Semester

Mathematics – Core

DIFFERENTIAL GEOMETRY

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. The line of intersection of the normal plane and the osculating plane at P is the ________ at P.

   (a) Tangent (b) Binormal

   (c) Principal normal (d) Rectifying plane

2. If the curvature $K = 0$ at all points of the curve, then the curve is a ________.

   (a) Helix (b) Circle

   (c) Bertrand curve (d) Straight line

3. The normal in a direction orthogonal to the osculating plane is ________ line.

   (a) Tangent (b) Osculating

   (c) Binormal (d) Involute

4. A point which is not an ordinary point is called a/an ________.

   (a) Essentiality (b) Singularity

   (c) Imaginary (d) Parametric

5. A surface generated by the screw motion of a curve about a fixed line is a ________.

   (a) Paraboloid (b) Sphere

   (c) Helicoid (d) Helix

6. If $ER - 2FQ + GP = 0$, then the two families are ________.

   (a) Parallel (b) Collinear

   (c) Isometric (d) Orthogonal

7. A curve is a geodesic if and only if at every point __________.

(a) Its principal normal is tangent to surface

(b) Its principal normal is normal to the surface

(c) The rectifying plane is tangent to surface

(d) The rectifying plane is parallel to the surface

8. When the geodesics are concurrent straight lines, the parallels are __________.

(a) Straight lines (b) Concentric circles

(c) Involutes (d) Helix

9. On any surface isometrix with the plane, the excess of a simple closed curve is __________.

(a) 2π (b) Zero

(c) π (d) $\frac{\pi}{2}$

10. A curve on a surface whose tangent at each point is along a principal direction is called __________.

(a) Line of curvature

(b) Gaussian curvature

(c) Principal curvature

(d) Asymptotic line

11. The edge of regression of the polar developable is the locus of __________ of the given curve.

(a) Centres of poles

(b) Centres of spherical curvature

(c) Mid points of tangents

(d) Mid points of normals

12. A ruled surface is generated by the motion of a straight line with __________ degree of freedom(s).

(a) Zero (b) Two

(c) One (d) Infinite

13. The only compact surfaces whose Gaussian curvature is positive and mean curvature constant are __________.

(a) Complete surface (b) Plane

(c) Analytic surfaces (d) Spheres

14. In a complete surface, every bounded set of points is __________.

(a) Umbilic (b) Singularity

(c) Relatively compact (d) Conjugate

15. The universal covering surface of a surface is always ___________.

(a) Compact (b) Simply connected

(c) Bounded (d) Hausdroff

PART B — ($5 \times 4 = 20$ marks)

Answer ALL questions choosing either (a) or (b).

16. (a) Show that $[\dot{r}, \ddot{r}, \dddot{r}] = 0$ is a necessary and sufficient condition that the curve be plane.

Or

(b) Find the equation of the osculating plane at a general point on the cubic curve given by $r = (u, u^2, u^3)$ and show that the osculating planes at any three points of the curve meet at a point lying in the plane determined by these 3 points.

17. (a) For the paraboloid $x = u$, $y = v$, $z = u^2 - v^2$, find $EG - F^2$ and hence H.

Or

(b) Find the coefficients of the direction which makes an angle $\frac{1}{2}\pi$ with the direction whose coefficients are (l, m).

18. (a) Prove that on the general surface, a necessary and sufficient condition that the curve $v = c$ be a geodesic is $EE_2 - FE_1 - 2EF_1 = 0$ when $v = c$, for all values of u.

Or

(b) Derive Liouville's formula for Kg.

19. (a) Show that if there is a surface of minimum area passing through a closed space curve, it is necessarily a minimal surface.

Or

(b) State Rodrigue's formula and prove it.

20. (a) Prove that the surface, in the neighbourhood of any point, is spherical or plane.

Or

(b) Give an example for a surface which is not complete.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions choosing either (a) or (b).

21. (a) Obtain the curvature and torsion of the curve of intersection of the two quadric surfaces $ax^2 + by^2 + cz^2 = 1$, $a'x^2 + b'y^2 + c'z^2 = 1$.

Or

(b) What are helices? Discuss its properties.

22. (a) When are two surfaces isometric? Find a surface of revolution which is isometric with a region of the right helicoid.

Or

(b) A helicoid is generated by the screw motion of a straight line which meets the axis at an angle α. Find the orthogonal trajectories of the generators. Find also the metric of the surface referred to the generators and their orthogonal trajectories as parametric curves.

23. (a) Using calculus of variations, find the equations for geodesics.

Or

(b) Show that two surfaces of the same constant curvature are locally isometric.

24. (a) Obtain a necessary and sufficient condition for a surface to be developable.

Or

(b) Show that the Gaussian curvature at the point distant v from the central point on a generator of parameter P is given by

$$K = \frac{-p^2}{\left(p^2 + v^2\right)^2}.$$

25. (a) In order that the geodesic distance AB should be the shortest distance, prove that it is necessary and sufficient to have B lying between A and its conjugate A.

Or

(b) On a complete surface, show that any two points can be joined by a geodesic arc whose length is equal to their distance.

(8 pages) **Reg. No. :** ....

Code No. : 5418 **Sub. Code : WMAE 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2025.

Fourth Semester

Mathematics

Elective VI – RING THEORY AND LATTICES

(For those who joined in July 2023-2024 only)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. The number of ideals of the set of all rational numbers is ________
   (a) 1
   (b) 2
   (c) 0
   (d) none of these

2. Suppose γ is a real number $0 \le \gamma \le 1$, $Mr = \{f(x) \in R \mid f(\gamma) = 0\}$ is a ________ ideal of R.
   (a) Left ideal (b) Right ideal
   (c) Prime ideal (d) Maximal ideal

3. If $M = < n >$ is a maximal ideal of R then n is a ________ number.
   (a) composite (b) prime
   (c) whole (d) natural

4. The number of units in the ring of integers is ________
   (a) 1 (b) 2
   (c) 0 (d) none of these

5. Let R is an Euclidean ring and $a, b \in R$. If $b \neq 0$ is not a unit in R then
   (a) $d(a) < d(ab)$ (b) $d(a) > d(ab)$
   (c) $d(a) = d(ab)$ (d) none of these

6. A solution of $x^2 \equiv -1 (\mathrm{mod}\, 13)$ is
   (a) 13 (b) 5
   (c) 6 (d) 11

7. If $f(x)=a_0+a_1x+...a_nx^n \neq 0$ and $a^n \neq 0$ then the degree of $f(x)$, written as degree $f(x)$ is

(a) $n-1$ (b) $n+1$

(c) 1 (d) n

8. If $f(x)$ and $g(x)$ are two polynomials then ————

(a) $def\ (f(x)\,g(x)) \leq \deg f(x),\ g(x) \neq 0$

(b) $def\ (f(x)\,g(x)) \geq \deg f(x),\ g(x) \neq 0$

(c) $def\ (f(x)\,g(x)) = \deg f(x).\ \text{degg}\ (x)\ g(x) \neq 0$

(d) $def\ (f(x)\,g(x)) = \deg f(x) - \text{degg}\ (x) g(x) \neq 0$

9. The degree of zero polynomial is

(a) 1 (b) not defined

(c) 3 (d) 0

10. The only idempotent element in rad R is ————

(a) 1 (b) 2

(c) 3 (d) 0

11. Let R be a principal ideal domain. Then R is semi-simple if and only R is

(a) a field

(b) has an infinite number of maximal ideals

(c) either a field or has an infinite number of maximal ideals

(d) none of the above

12. The only idempotent element in rad R is

(a) 0 (b) 1

(c) 2 (d) both (a) and (b)

13. A 1 –to –1 mapping a to a' of a lattice L onto a lattice L' is an homomorphism if

(a) $a \cap b = (a \cap b)'$

(b) $(a \cup b)' = a' \cup b'$

(c) $(a \cup b)' = a' \cap b'$

(d) All the above

14. Which of the following is non complete lattice?

(a) Z (b) Q

(c) W (d) N

15. Let B be a Boolean Algebra and $a.b \in B$. Then $(a \cap b)' \cup (a' \cap b)$ is

(a) $a-b$ (b) $a+b$

(c) ab (d) $b-a$

PART B — $(5 \times 4 = 20$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that if $[a,b]=[a',b']$ and $[c,d]= [c',d']$ then $[a,b][c,d]=[a',b'][c',d']$.

Or

(b) If ϕ is a homomorphism of R into R^ with Kernel $I(\phi)$ then prove the following

(i) If ϕ is a sub group of R under addition

(ii) If $\alpha \in I(\phi)$ and $r \in R$ then both ar and ra are in $I(\phi)$.

17. (a) Prove that a Euclidean ring posses a unit element.

Or

(b) Let R be a Euclidean ring. Then show that any two elements a and b in R has a greatest common divisor d Moreover $d=\lambda a+\mu b$ for same $\lambda, \mu \in R$.

18. (a) State and prove the Gauss lemma.

Or

(b) Define primitive polynomial and prove that if $f(x)$ and $g(x)$ are primitive polynomials, then $f(x)g(x)$ is a primitive polynomial.

19. (a) Let I be an ideal of R then prove that $I \subseteq$ rad R if and only if each element of the coset $1+I$ has an inverse in G.

Or

(b) For any ring R, prove that the quotient ring $R/$ Rad R is without prime radical.

20. (a) Prove that a lattice L is modular if and only if $a \geq b$ and $a \cup c = b \cup c, a \cap c = b \cap c$ for any c implies that $a=b$.

Or

(b) Prove that the complement a' of any element a of a Boolean algebra B is uniquely determined. The mapping a to a is 1-1 o B onto itself, it is of period 2.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

21. (a) If R is a commutative ring with unit element and M is an ideal of R then show that M is a maximal ideal of R if and only if R/M is a field.

Or

(b) Let R and R' be rings and ϕ a homomorphism of R onto R' with kernel U. Then prove that R' is isomorphic to R/U. Moreover there is a correspondence between the set of ideals of R' and the set of ideals of R which contain U. This correspondence can be achieved by associating with an ideal W' in R' the ideal W in R defined by $W = \{x \varepsilon R / \phi(x) \varepsilon W'\}$ with W so defined R/W is isomorphic to R'/W'.

22. (a) State and prove Fermat's theorem.

Or

(b) If π is a prime element in the Euclidean ring R and π / ab where $a, b \varepsilon R$, then prove that π divides at least one of a or b.

23. (a) State and prove the Eisenstein criterion.

Or

(b) Define unique factorization domain and prove that if R is a unique factorization domain then so is $R[x_1, x_2, \ldots x_n]$.

24. (a) Let I be an ideal of the ring R. Further, assume that the subset $S \subseteq R$ is closed under Multiplication and disjoint from I. Then prove that there exits an ideal P which is maximal in the set of ideals which contain I and do not meet S; any such ideal is necessarily prime.

Or

(b) If I is an ideal of the ring R, then

(i) $rad\,(R/I) \supseteq (rad\,R + 1)/I$

(ii) whenever $I \subseteq rad\,R$, $rad\,(R/I) = (rad\,R)/I$.

25. (a) Any two finite descending chain connecting the elements a, b of a modular lattice have equivalence refinements.

Or

(b) State and prove Stone's theorem.

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